In the history of mathematics, the concept of infinitesimal is the cornerstone of calculus. However the development of this crucial concept is full of twists and turns. In this paper, I want to review some conceptual history of infinitesimal and compare the view of Isaac Newton, George Berkeley, and Leonhard Euler, and finally I would also discuss the modern construction of non-standard analysis where we can rigorously treat infinitesimal and infinity as numbers.

At the infant era of calculus, people tried to abstract the concept of infinitesimal from physical phenomenon like motion of objects, where infinitesimals like $dx$ $dt$ are some small quantities that differs from 0 but closer to $0$ than any other numbers. Newton developed the method of fluxion to calculate momentary velocity. For example, how did he compute the momentary velocity of a particle, where the position $x$ relies on time $t$ as

$$

x=t^n?

$$

1. Consider $t$ increased by an infinitesimal amount $o$

$$

x+o\cdot \dot{x}=(t+o)^n,

$$

where $\dot(x)$ is called fluxion, which amounts to derivative of $x$ with respect to $t$ in modern language.

2. Then take the difference of the new equation and the original equation, we get

$$

o\cdot\dot{x}=o\cdot mx^{m-1}+\frac{m(m-1)}{2}\cdot o^2 x^{m-2}+…+a o^m.

$$

3. And then we divide both sides by $o$, notice that this step requires $o$ to be non-zero.

$$

\dot(x)=m x^{m-1}+o\cdot (….)

$$

4. Finally abandon all the terms on the right hand side containing $o$.

The key point (joke) is to regard the infinitesimal $o$ as a nonzero small value when it is in the denominator and abandon it whenever we want. Of course it is logically inconsistent, but Newton had captured the most important feature of infinitesimal, and the above procedure is a efficient algorithm to calculate simple derivatives.

In his monograph “The Analyst: a discourse addressed to an infidel mathematician”, George Berkeley observed the logical fallacy and criticized:

“They are neither finite quantities not quantities infinitely small, nor yet nothing. May we not call them the ghost of departed quantities?”

Berkeley observed the above calculation can be identified as the following steps:

1. expand the expression
2. take the difference
3. divide by a common $o$
4. replace $o$ with $0$.

The third step and the forth step are not “commutative”, since if we change the order, the calculation if meanless.